

# WHEN DIFFERENT ENTANGLEMENT WITNESSES DETECT ENTANGLED STATES SIMULTANEOUSLY

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**ABSTRACT.** The question under what conditions different witnesses may detect some entangled states simultaneously is answered for both finite- and infinite-dimensional bipartite systems. Finite many different witnesses can detect some entangled states simultaneously if and only if their convex combinations are still witnesses; they can not detect any entangled state simultaneously if and only if the set of their convex combinations contains a positive operator. For two witnesses  $W_1$  and  $W_2$ , some more can be said: (1)  $W_1$  and  $W_2$  can detect the same set of entangled states if and only if they are linearly dependent; (2)  $W_2$  can detect more entangled states than that  $W_1$  can if and only if  $W_1$  is a linear combination of  $W_2$  and a positive operator. As an application, some characterizations of the optimal witnesses are given and some structure properties of the decomposable optimal witnesses are presented.

## 1. INTRODUCTION

The challenging question of characterizing the quantum entangled states has attracted much attention in recent years. However, despite remarkable progress in this field, there is no general qualitative and quantitative characterizing of entanglement [3, 4, 5, 6, 8, 9, 10, 11, 15, 18, 19].

Recall that, a bipartite quantum state (or density operator) in an bipartite system is a positive trace one operator  $\rho$  (i.e.  $\rho \geq 0$  and  $\text{Tr}(\rho) = 1$ ) acting on a complex tensor product Hilbert space  $H_1 \otimes H_2$  which describing the bipartite quantum system, where  $H_1$  and  $H_2$  are complex separable Hilbert spaces describing the corresponding subsystems (we also say a unit vector in the corresponding Hilbert space is a pure state). If both  $H_1$  and  $H_2$  are finite-dimensional, then the composite system is a finite-dimensional system; if at least one of  $H_1$  and  $H_2$  is infinite-dimensional, then the composite system is an infinite-dimensional system. By  $\mathcal{S}^{(1)} = \mathcal{S}(H_1)$ ,  $\mathcal{S}^{(2)} = \mathcal{S}(H_2)$  and  $\mathcal{S} = \mathcal{S}(H_1 \otimes H_2)$  we denote the sets of all states on  $H_1$ ,  $H_2$  and  $H_1 \otimes H_2$ , respectively. Note that, when  $\dim H_1 \otimes H_2 = \infty$ , we have  $\mathcal{S} \subset \mathcal{T}(H_1 \otimes H_2)$ , the Banach space of all trace-class operators on  $H_1 \otimes H_2$  with trace norm  $\|\cdot\|_{\text{Tr}}$ . A state  $\rho \in \mathcal{S}$  is

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said to be *separable* if it is a trace-norm limit of the states of the form

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)},$$

where  $\rho_i^{(1)}$  and  $\rho_i^{(2)}$  are pure states in  $\mathcal{S}^{(1)}$  and  $\mathcal{S}^{(2)}$ , respectively,  $\sum_i p_i = 1$ ,  $p_i \geq 0$ . Otherwise,  $\rho$  is said to be *entangled* (or inseparable). The set of all separable states will be denoted by  $\mathcal{S}_{\text{sep}}(H_1 \otimes H_2)$ .

Among the multitudinous criteria for deciding whether a given state is entangled or not, the well known one is the *entanglement witness criterion* [8]. This criterion provides a sufficient and necessary condition for separability of a given state in a bipartite quantum system. It is shown that [8], a given state is separable if and only if there exists at least one entanglement witness detecting it. A self-adjoint operator (also called hermitian operator some times)  $W$  acting on  $H_1 \otimes H_2$  is called an *entanglement witness* (or *witness* for short) if  $\text{Tr}(W\sigma) \geq 0$  for all separable sates  $\sigma \in \mathcal{S}_{\text{sep}}$  and  $\text{Tr}(W\rho) < 0$  for at least one entangled state  $\rho$  (in this case, we say that  $\rho$  is detected by  $W$ , or, equivalently,  $W$  is a witness for  $\rho$ ).

Although any entangled state can be detected by some specific choice of witness, there is no universal witness, i.e., there is no witness which can detect all entangled states. From the entanglement witness criterion, the task is reduced to find out all witnesses. However, constructing the witnesses for an entangled state is a hard task, and the determination of witnesses for all entangled states is a NP-hard problem [1].

Witnesses not only can be used to detect any entangled states, but also are directly measurable quantities. This makes the entanglement witnesses one of the main methods to detect entanglement experimentally and a very useful tool for analyzing entanglement in experiment. So, it is important to know more about the features of the witnesses. Concerning this topic, much work has been done for finite-dimensional systems (for example, ref. [16, 21]). However, few results are known for infinite-dimensional systems. Generally, the structure of witnesses for infinite-dimensional systems are complicated. However, it was proved in [12] that, for any entangled state, a witness can be chosen so that it has a simple form of “nonnegative constant times identity + a self-adjoint operator of finite rank”. This kind of witnesses are Fradholm operator of index 0 with the spectrum consisting of finite many eigenvalues and hence are easily handled. The goal of the present paper is to solve the question when deferent witnesses can detect some entangled states simultaneously for mainly infinite-dimensional systems.

For simplicity, we introduce some notations. Let  $H_1, H_2$  be complex Hilbert spaces and let  $\mathcal{W} = \mathcal{W}(H_1 \otimes H_2)$  be the set of all entanglement witnesses of the system  $H_1 \otimes H_2$ , i.e.,

$$\begin{aligned}\mathcal{W} &= \mathcal{W}(H_1 \otimes H_2) \\ &= \{W : W \in \mathcal{B}(H_1 \otimes H_2), W^\dagger = W, \\ &\quad \text{Tr}(W\sigma) \geq 0 \text{ for all } \sigma \in \mathcal{S}_{\text{sep}} \text{ and } W \text{ is not positive}\}.\end{aligned}$$

For  $W \in \mathcal{W}$  and  $\Gamma \subset \mathcal{W}$ , define

$$\mathcal{D}_W = \{\rho : \rho \in \mathcal{S}, \text{Tr}(W\rho) < 0\}$$

and  $\mathcal{D}_\Gamma = \bigcap_{W \in \Gamma} \mathcal{D}_W$ . Then  $\mathcal{D}_W$  and  $\mathcal{D}_\Gamma$  are convex sets. Thus the witnesses in  $\Gamma$  can detect some entangled states simultaneously if and only if  $\mathcal{D}_\Gamma \neq \emptyset$ .

For  $W_1, W_2 \in \mathcal{W}$ , if  $\mathcal{D}_{W_2} \subset \mathcal{D}_{W_1}$ , we say that  $W_1$  is *finer* than  $W_2$ , denoted by

$$W_2 < W_1.$$

We call  $W_1$  is an *optimal* witness if there exists no other witness finer than  $W_1$ . Then  $\mathcal{W}$  becomes a poset with respect to the partial order “ $<$ ”. Generally, for two given witnesses  $W_1$  and  $W_2$ , there are three different situations that may happen: (i)  $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$ , i.e.,  $W_1 < W_2$ , in particular,  $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$ ; (ii)  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$  and  $\mathcal{D}_{W_i} \not\subseteq \mathcal{D}_{W_j}$ ,  $i, j = 1, 2$ ; (iii)  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$ . Thus  $W_1$  and  $W_2$  can detect a state simultaneously if and only if (i) or (ii) holds.

For the finite-dimensional case, the relations (i)-(iii) above are studied in [16, 21]. Suppose that  $\text{Tr}(W_1) = \text{Tr}(W_2)$ , then the following conclusions are true: (i)  $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$  if and only if  $W_1 = (1 - \varepsilon)W_2 + \varepsilon D$  for some  $D \geq 0$  and  $0 \leq \varepsilon < 1$ ; in particular,  $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$  if and only if  $W_1 = W_2$  [16]; (ii) if there are no inclusion relations between  $\mathcal{D}_{W_1}$  and  $\mathcal{D}_{W_2}$ , then  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$  if and only if  $W = \epsilon W_1 + (1 - \epsilon)W_2$  is not positive for any  $0 \leq \epsilon \leq 1$  [21]. However, there are some mistakes in the proof of [21]. The main purpose of the present paper is to show that the similar results holds for infinite-dimensional systems and to correct the mistakes appeared in [21]. Note that, the condition  $\text{Tr}(W_1) = \text{Tr}(W_2)$  makes no sense in general for infinite-dimensional case.

This paper is organized as follows. In Section 2, we propose a sufficient and necessary condition for any two given general witness  $W_1$  and  $W_2$  to satisfy  $W_1 < W_2$ . Let  $H_1, H_2$  be complex Hilbert spaces. Assume that  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ . We show that, (1)  $W_1 < W_2$  if and only if  $W_1 = aW_2 + D$  for some operator  $D \geq 0$  and some scalar  $a > 0$ ; (2)  $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$  if and only if there exists a positive number  $a > 0$  such that  $W_1 = aW_2$ . Then these results are applied in Section 3 to obtain a sufficient and necessary condition for a witness to be optimal. We show that  $W \in \mathcal{W}(H_1 \otimes H_2)$  is optimal if and only if for any nonzero operator  $D \geq 0$  and scalar  $a > 0$ ,  $W' = aW - D \notin \mathcal{W}(H_1 \otimes H_2)$ . Some structure properties of the optimal decomposable witnesses are also presented. In Section 4, we discuss the question when finite many witnesses can detect a common entangled state. We show that  $\bigcap_{k=1}^n \mathcal{D}_{W_k} \neq \emptyset$  if and only

if every combination of  $W_1, \dots, W_n$  is still a witness;  $\cap_{k=1}^n \mathcal{D}_{W_k} = \emptyset$  if and only if there exists at least one convex combination  $W$  of  $W_1, \dots, W_n$  such that  $W \geq 0$ .

Throughout this paper, we call an operator  $A \in \mathcal{B}(H)$  is positive, if  $\langle x|A|x \rangle \geq 0$  for all  $|x\rangle \in H$ .  $\|\cdot\|_{\text{Tr}}$  denotes the trace norm, and  $\|\cdot\|_2$  denotes the Hilbert-Schmidt norm. For a operator  $A$ ,  $A^T$  stands for the transposition of  $A$  with respect some given orthonormal basis. By  $A^{T_2}$  we denote the partial transposition of  $A$  with respect to the second subsystem  $H_2$ , i.e.,  $A^{T_2} = (I_1 \otimes \tau)A$ , where  $\tau$  is the transpose operation.  $\mathcal{T}(H_1 \otimes H_2)$  denotes the set of all trace class operators in  $\mathcal{B}(H_1 \otimes H_2)$  while  $\mathcal{T}^+(H_1 \otimes H_2)$  stands for the set of all positive elements in  $\mathcal{T}(H_1 \otimes H_2)$ .

## 2. WITNESSES WITH THE FINER RELATION BETWEEN THEM

In this section, we mainly highlight the *finer* relation between two given general witnesses of an *infinite-dimensional* bipartite system.

For finite-dimensional bipartite quantum system, it is known that if  $W_1, W_2 \in \mathcal{W}$  with  $\text{Tr}(W_1) = \text{Tr}(W_2)$ , then  $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$  if and only if  $W_1 = (1 - \varepsilon)W_2 + \varepsilon D$  for some  $D \geq 0$  and  $0 \leq \varepsilon < 1$ ;  $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$  if and only if  $W_1 = W_2$  [16]. Since the condition  $\text{Tr}(W_1) = \text{Tr}(W_2)$  makes no sense in general for infinite-dimensional case, we have to consider the question without the trace-equal assumption.

The following is the main result in this section which answers the question when  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \mathcal{D}_{W_1}$  for both infinite-dimensional systems and finite-dimensional cases.

**Theorem 2.1.** *Let  $H_1, H_2$  be complex Hilbert spaces. Assume that  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ . Then*

- (1)  $W_1 < W_2$  if and only if  $W_1 = aW_2 + D$  for some operator  $D \geq 0$  and some scalar  $a > 0$ .
- (2)  $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$  if and only if there exists a positive number  $a > 0$  such that  $W_1 = aW_2$ .

To prove Theorem 2.1, we need several lemmas.

We first generalize a useful result in [16] to infinite-dimensional case, which asserts that the restriction of any entanglement witness as a linear functional to the convex set consisting of separable states is nonzero.

**Lemma 2.2.** *Let  $H_1, H_2$  be complex Hilbert spaces. For any  $W \in \mathcal{W}(H_1 \otimes H_2)$ , there is a separable pure state  $\sigma \in \mathcal{S}_{\text{sep}}(H_1 \otimes H_2)$  such that  $\text{Tr}(W\sigma) > 0$ .*

**Proof.** Let  $\{|i\rangle\}$  and  $\{|j\rangle\}$  be any orthonormal bases of  $H_1$  and  $H_2$ , respectively. Then,  $\{|i\rangle|j\rangle\}$  is an orthonormal basis of  $H_1 \otimes H_2$ . It turns out  $\langle i|\langle j|W|i\rangle|j\rangle \geq 0$  since  $\langle i|\langle j|W|i\rangle|j\rangle = \text{Tr}(W|i\rangle\langle i| \otimes |j\rangle\langle j|) \geq 0$  for any  $i, j$ .

To prove the lemma, it is suffice to show that there exist orthonormal bases  $\{|i\rangle\}$  and  $\{|j\rangle\}$  such that  $\text{Tr}(W|i\rangle\langle i| \otimes |j\rangle\langle j|) \neq 0$  for some  $i, j$ . To get a contradiction, assume that this is not true. Then

$$\langle \psi_1 | \langle \psi_2 | W | \psi_1 \rangle | \psi_2 \rangle = 0$$

for all product vectors  $|\psi_1\rangle|\psi_2\rangle \in H_1 \otimes H_2$ . For any pure state  $|\psi\rangle \in H_1 \otimes H_2$ , let  $|\psi\rangle = \sum_{k=1}^n \lambda_k |k\rangle|k'\rangle$  be the Schmidt decomposition of  $|\psi\rangle$ , where  $\lambda_k > 0$ ,  $\sum_{k=1}^n \lambda_k^2 = 1$  and  $\{|k\rangle\}_{k=1}^n, \{|k'\rangle\}_{k'=1}^n$  are orthonormal sets respectively in  $H_1, H_2$ , here  $n$  is called the Schmidt number of  $|\psi\rangle$ . Then,

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| \\ &= (\sum_k \lambda_k |k\rangle|k'\rangle)(\sum_l \lambda_l \langle l|\langle l'|) \\ &= \sum_{k,l} \lambda_k \lambda_l |k\rangle\langle l| \otimes |k'\rangle\langle l'| \\ &= \sum_{k=l} \lambda_k^2 |k\rangle\langle k| \otimes |k'\rangle\langle k'| + \sum_{k < l} \lambda_k \lambda_l (|k\rangle\langle l| \otimes |k'\rangle\langle l'| + |l\rangle\langle k| \otimes |l'\rangle\langle k'|). \end{aligned}$$

For given pair  $(k, l)$  with  $k \neq l$ , define  $|\psi_{k,l}\rangle = \frac{1}{\sqrt{2}}(|k\rangle|k'\rangle + |l\rangle|l'\rangle)$ . We have

$$|k\rangle\langle l| \otimes |k'\rangle\langle l'| + |l\rangle\langle k| \otimes |l'\rangle\langle k'| = 2|\psi_{k,l}\rangle\langle\psi_{k,l}| - |k\rangle\langle k| \otimes |k'\rangle\langle k'| - |l\rangle\langle l| \otimes |l'\rangle\langle l'|.$$

This indicates that, if  $n < \infty$ , then  $\langle\psi|W|\psi\rangle = 0$ . As the set of all unit vectors with the finite Schmidt number is dense in the set of all unit vectors in  $H_1 \otimes H_2$ , we see that  $\langle\psi|W|\psi\rangle = 0$  holds for all unit vector  $|\psi\rangle$  and hence  $W = 0$ , a contradiction.  $\square$

Analogues to the finite-dimensional case [16], the following lemma is obvious.

**Lemma 2.3.** *Let  $H_1, H_2$  be complex Hilbert spaces. For a given  $W \in \mathcal{W}(H_1 \otimes H_2)$ , if  $\rho \in \mathcal{D}_W$  and  $\varrho_W \in \mathcal{T}^+(H_1 \otimes H_2)$  satisfying  $\text{Tr}(W\varrho_W) = 0$ , then  $(\rho + \varrho_W)/\text{Tr}(\rho + \varrho_W) \in \mathcal{D}_W$ .*

The next lemma is crucial for our purpose. Its statement as well as its proof are quite different from the counterpart lemma in [16] for finite-dimensional case.

**Lemma 2.4.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ . Assume that  $W_1 < W_2$  and let*

$$\lambda := \inf_{\rho_1 \in \mathcal{D}_{W_1}} \frac{|\text{Tr}(W_2 \rho_1)|}{|\text{Tr}(W_1 \rho_1)|}.$$

*Then the following statements are true:*

- (1) *If  $\rho \in \mathcal{S}(H_1 \otimes H_2)$  satisfies  $\text{Tr}(W_1 \rho) = 0$ , then  $\text{Tr}(W_2 \rho) \leq 0$ ;*
- (2)  $\lambda > 0$ .
- (3) *If  $\rho \in \mathcal{S}(H_1 \otimes H_2)$  satisfies  $\text{Tr}(W_1 \rho) > 0$ , then  $\text{Tr}(W_2 \rho) \leq \lambda \text{Tr}(W_1 \rho)$ .*

**Proof.** (1) Let us assume, to reach a contradiction, that  $\text{Tr}(W_2 \rho) > 0$ . Then, for any  $\rho_1 \in \mathcal{D}_{W_1}$  and  $a \geq 0$ , we have  $\rho(a) = (\rho_1 + a\rho)/(1+a) \in \mathcal{D}_{W_1}$ . On the other hand, there exists a positive number  $a_0$  such that  $\text{Tr}(W_2 \rho(a)) > 0$  holds for all  $a \geq a_0$ , which is impossible since it leads to  $\rho(a) \notin \mathcal{D}_{W_2}$ .

(2) Assume that, on the contrary,  $\lambda = 0$ . Then, there exists a sequence  $\{\rho_n\} \subset \mathcal{D}_{W_1}$  such that

$$\varepsilon_n = \frac{\text{Tr}(W_2 \rho_n)}{\text{Tr}(W_1 \rho_n)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.1)$$

Note that there exists  $\sigma \in \mathcal{S}_{\text{sep}} = \mathcal{S}_{\text{sep}}(H_1 \otimes H_2)$  such that both  $\text{Tr}(W_1 \sigma)$  and  $\text{Tr}(W_2 \sigma)$  are nonzero. If not, then for any  $\sigma \in \mathcal{S}_{\text{sep}}$ , either  $\text{Tr}(W_1 \sigma) = 0$  or  $\text{Tr}(W_2 \sigma) = 0$ . Thus, by Lemma 2.2, there exist  $\sigma_1, \sigma_2 \in \mathcal{S}_{\text{sep}}$  so that  $\text{Tr}(W_1 \sigma_1) = t > 0$ ,  $\text{Tr}(W_1 \sigma_2) = 0$ ,  $\text{Tr}(W_2 \sigma_1) = 0$  and

$\text{Tr}(W_2\sigma_2) = s > 0$ . Let  $\sigma = \frac{s}{t+s}\sigma_1 + \frac{t}{t+s}\sigma_2 \in \mathcal{S}_{\text{sep}}$ . Then  $\text{Tr}(W_1\sigma) = \text{Tr}(W_2\sigma) = \frac{ts}{t+s} \neq 0$ , contradicting to the assumption.

Now we can take  $\sigma \in \mathcal{S}_{\text{sep}}$  so that both  $\text{Tr}(W_1\sigma)$  and  $\text{Tr}(W_2\sigma)$  are nonzero. Let

$$\tilde{\rho}_n = \frac{1}{1 - \frac{\text{Tr}(W_1\rho_n)}{\text{Tr}(W_1\sigma)}}(\rho_n - \frac{\text{Tr}(W_1\rho_n)}{\text{Tr}(W_1\sigma)}\sigma) \in \mathcal{S}$$

with  $\rho_n$  satisfying Eq.(2.1). Then  $\text{Tr}(W_1\tilde{\rho}_n) = 0$  and by (1), we have  $\text{Tr}(W_2\tilde{\rho}_n) \leq 0$  for every  $n$ . However,

$$\begin{aligned} \text{Tr}(W_2\tilde{\rho}_n) &= \frac{1}{1 - \frac{\text{Tr}(W_1\rho_n)}{\text{Tr}(W_1\sigma)}}(\text{Tr}(W_2\rho_n) - \frac{\text{Tr}(W_1\rho_n)}{\text{Tr}(W_1\sigma)}\text{Tr}(W_2\sigma)) \\ &= \frac{1}{1 - \frac{\text{Tr}(W_1\rho_n)}{\text{Tr}(W_1\sigma)}}(\varepsilon_n - \frac{\text{Tr}(W_2\sigma)}{\text{Tr}(W_1\sigma)}\text{Tr}(W_1\rho_n)) \end{aligned}$$

and  $\varepsilon_n \rightarrow 0$ , which implies that for sufficient large  $n$ , we have  $\varepsilon_n - \frac{\text{Tr}(W_2\sigma)}{\text{Tr}(W_1\sigma)} < 0$  and hence  $\text{Tr}(W_2\tilde{\rho}_n) > 0$ , a contradiction. This completes the proof of (2).

(3) Assume that  $\text{Tr}(W_1\rho) > 0$ . Take  $\rho_1 \in \mathcal{D}_{W_1}$  and let  $\tilde{\rho} = \frac{1}{\text{Tr}(W_1\rho) - \text{Tr}(W_1\rho_1)}[\text{Tr}(W_1\rho)\rho_1 - \text{Tr}(W_1\rho_1)\rho]$ . Then we have  $\text{Tr}(W_1\tilde{\rho}) = 0$ . By (1), we obtain that  $\text{Tr}(W_2\tilde{\rho}) \leq 0$ . Thus we have  $\text{Tr}(W_1\rho)\text{Tr}(W_2\rho_1) \leq \text{Tr}(W_1\rho_1)\text{Tr}(W_2\rho)$ . It follows that

$$\frac{\text{Tr}(W_2\rho)}{\text{Tr}(W_1\rho)} \leq \frac{|\text{Tr}(W_2\rho_1)|}{|\text{Tr}(W_1\rho_1)|}.$$

Taking the infimum with respect to  $\rho_1 \in \mathcal{D}_{W_1}$  on the right side of the above equation, we get  $\text{Tr}(W_2\rho) \leq \lambda\text{Tr}(W_1\rho)$ .  $\square$

Now we are in a position to give our proof of Theorem 2.1.

**Proof of Theorem 2.1.** (1) If  $W_1 = aW_2 + D$  for some positive operator  $D$  and some scalar  $a > 0$ , then for any  $\rho \in \mathcal{D}_{W_1}$ , we have  $a\text{Tr}(W_2\rho) + \text{Tr}(D\rho) = \text{Tr}(W_1\rho) < 0$ , which implies that  $\text{Tr}(W_2\rho) < 0$ . Hence  $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$ . Conversely, assume that  $\mathcal{D}_{W_1} \subseteq \mathcal{D}_{W_2}$ . Then, by Lemma 2.4,

$$\text{Tr}(W_2\rho) \leq \lambda\text{Tr}(W_1\rho) \tag{2.2}$$

holds for all  $\rho \in \mathcal{S}$ , where  $\lambda = \inf_{\rho_1 \in \mathcal{D}_{W_1}} \frac{|\text{Tr}(W_2\rho_1)|}{|\text{Tr}(W_1\rho_1)|} > 0$ . This implies that  $D_1 = \lambda W_1 - W_2 \geq 0$  and hence, with  $D = \lambda^{-1}D_1$ ,  $W_1 = \lambda^{-1}W_2 + D$ , as desired.

(2) We only need to prove the ‘only if’ part. Assume that  $\mathcal{D}_{W_1} = \mathcal{D}_{W_2}$ . Then, by the statement (1) just proved above, there exist operators  $D_i \geq 0$  and scalars  $a_i > 0$ ,  $i = 1, 2$ , such that  $W_1 = a_1W_2 + D_1$  and  $W_2 = a_2W_1 + D_2$ . It follows that  $W_1 = a_1(a_2W_1 + D_2) + D_1 = a_1a_2W_1 + a_1D_2 + D_1$ . Thus  $(1 - a_1a_2)W_1 = a_1D_2 + D_1 \geq 0$ . Since  $W_1 \in \mathcal{W}$ , we must have  $a_1a_2 = 1$ . Hence  $D_1 = D_2 = 0$  and  $W_2 = a_2W_1$ , completing the proof.  $\square$

### 3. OPTIMIZATION OF ENTANGLEMENT WITNESSES

In this section we discuss the optimization of entanglement witnesses, especially for infinite-dimensional systems by applying Theorem 2.1.

The following result states that a witness is optimal if and only if any negative permutation if it will break the witness. For finite-dimensional case, a similar result was obtained in [16].

**Theorem 3.1.** *Let  $H_1, H_2$  be complex Hilbert spaces. Then  $W \in \mathcal{W}(H_1 \otimes H_2)$  is optimal if and only if for any nonzero operator  $D \geq 0$  and scalar  $a > 0$ ,  $W' = aW - D \notin \mathcal{W}(H_1 \otimes H_2)$ .*

**Proof.** To prove the ‘if’ part, assume, on the contrary, that  $W$  is not optimal, then  $W < W'$  for some  $W' \in \mathcal{W}(H_1 \otimes H_2)$  with  $W$  and  $W'$  are linearly independent. It follows from Theorem 2.1(1) that  $W = aW' + D$  for some  $D \geq 0$  and  $a > 0$ , which reveals that  $W' = \frac{1}{a}W - \frac{1}{a}D$ .

To prove the ‘only if’ part, assume that  $W$  is optimal but there exist nonzero operator  $D \geq 0$ , scalar  $a > 0$  so that  $W' = aW - D \in \mathcal{W}(H_1 \otimes H_2)$ . Then  $W = \frac{1}{a}W' + \frac{1}{a}D$  and  $W'$  is linearly independent to  $W$ . But by Theorem 2.1,  $W < W'$ , a contradiction.  $\square$

In the following, we discuss the condition for an entanglement witness that it cannot subtract some positive operators. For convenience, we define

$$\mathcal{P}_W = \{ |\psi\rangle\langle\phi| \in H_1 \otimes H_2 : \langle\psi|\langle\phi|W|\psi\rangle|\phi\rangle = 0 \}. \quad (3.1)$$

**Proposition 3.2.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W \in \mathcal{W}(H_1 \otimes H_2)$ . Let  $\mathcal{P}_W$  be as in Eq.(3.1). If  $D \in \mathcal{B}(H_1 \otimes H_2)$  is positive and  $D\mathcal{P}_W \neq \{0\}$ , then  $W - aD \notin \mathcal{W}(H_1 \otimes H_2)$  for any  $a > 0$ .*

**Proof.** If  $D\mathcal{P}_W \neq \{0\}$ , then there exists a product vector  $|\psi_0\rangle\langle\phi_0| \in \mathcal{P}_W$  such that

$$\langle\psi_0|\langle\phi_0|D|\psi_0\rangle|\phi_0\rangle > 0.$$

Write  $\rho_0 = |\psi_0\rangle\langle\psi_0| \otimes |\phi_0\rangle\langle\phi_0|$ . It is clear that  $\text{Tr}[(W - aD)\rho_0] = -a\text{Tr}(D\rho_0) < 0$ , which leads to  $W - aD \notin \mathcal{W}(H_1 \otimes H_2)$  for all  $a > 0$ .  $\square$

The following corollary is obvious.

**Corollary 3.3.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W \in \mathcal{W}(H_1 \otimes H_2)$ . Let  $\mathcal{P}_W$  be as in Eq.(3.1). If  $\mathcal{P}_W$  spans  $H_1 \otimes H_2$ , then  $W$  is optimal.*

Next we give some structure properties of optimal decomposable witnesses. Recall that a self-adjoint operator  $A \in \mathcal{B}(H_1 \otimes H_2)$  is said to be *decomposable* if

$$A = P + Q^{T_2}$$

for some operators  $P \geq 0, Q \geq 0$ , where  $Q^{T_2}$  denotes the partial transpose of  $Q$  with respect to the second subsystem  $H_2$ . Otherwise,  $A$  is said to be *indecomposable*. For example, in  $n \times n$  system, the Hermitian swap operator  $V = \sum_{i,j=0}^{n-1} |i\rangle\langle j| \otimes |j\rangle\langle i|$  is a decomposable witness since: (1)  $\text{Tr}(V\sigma) \geq 0$  for all separable pure states  $\sigma$ ; (2)  $V$  has a negative eigenvalue -1; (3)  $V = nQ^{T_2}$  with  $Q = |\psi\rangle\langle\psi|$  with  $|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} |i\rangle|i\rangle$  (ref. [20]). The examples of indecomposable witnesses can be found in [2, 7, 12]. It is easy to show that the decomposable witnesses can not detect any PPT entangled states (PPT stands for *positive partial transposition* as usual, [14]).

By applying Theorem 2.1, one can get a simple structure property of optimal decomposable entanglement witnesses for both finite-dimensional systems and infinite-dimensional systems.

**Theorem 3.4.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W \in \mathcal{W}(H_1 \otimes H_2)$  be a decomposable entanglement witness. If  $W$  is optimal, then  $W = Q^{T_2}$  for some  $Q \geq 0$ , and  $Q$  contains no product vectors in its range.*

**Proof.** Since  $W$  is decomposable, so  $W = P + Q^{T_2}$  for some positive operators  $P, Q$ . Assume that  $P \neq 0$ . As  $\text{Tr}(Q^{T_2}\sigma) = \text{Tr}(Q\sigma^{T_2}) \geq 0$  for all  $\sigma \in \mathcal{S}_{\text{sep}}$  and  $W \in \mathcal{W}$ , we must have  $Q^{T_2} \in \mathcal{W}$ . Thus, by Theorem 2.1 (1), one sees that  $W < Q^{T_2}$ , that is,  $W$  is not optimal. Hence,  $W$  is optimal implies that  $P = 0$  and  $W = Q^{T_2}$ . Moreover, the range of  $Q$  contains no product vectors. In fact, if  $|\psi\rangle|\phi\rangle \in R(Q)$  for some unit vectors  $|\psi\rangle \in H_1$  and  $|\phi\rangle \in H_2$ , then there exists a vector  $|\omega\rangle \in H_1 \otimes H_2$  such that  $Q|\omega\rangle = |\psi\rangle \otimes |\phi\rangle$ . Observe that  $Q(I - \lambda|\omega\rangle\langle\omega|)Q = Q^2 - \lambda|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| \geq 0$  if and only if  $I - \lambda|\omega\rangle\langle\omega| \geq 0$ . It turns out that, for any  $0 < \lambda < \|\omega\|^{-2}$  we have  $[Q - \lambda|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|]^{T_2} \in \mathcal{W}$ , which implies that  $[Q - \lambda|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|]^{T_2}$  is finer than  $W$ , contradicting to the optimality of  $W$ .  $\square$

Theorem 3.4 can be strengthened a little.

**Theorem 3.5.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W \in \mathcal{W}(H_1 \otimes H_2)$  be a decomposable entanglement witness. If  $W$  is optimal, then  $W = Q^{T_2}$  for some  $Q \geq 0$  and there exists no positive operator  $A$  with  $R(A) \subseteq R(Q)$  such that  $A^{T_2} \geq 0$ .*

**Proof.** By Theorem 3.4,  $W = Q^{T_2}$  as  $W$  is optimal. If there exists a positive operator  $A$  such that  $R(A) \subseteq R(Q)$  and  $A^{T_2} \geq 0$ , then, by a well known result from operator theory, there exists an operator  $T \in \mathcal{B}(H_1 \otimes H_2)$  such that  $A = QT$ . It follows that  $A^2 = QTT^\dagger Q \leq tQ^2$ , where  $t = \|T\|^2$ . Thus,  $A \leq \sqrt{t}Q$ , which implies  $Q - \lambda A \geq 0$  whenever  $0 < \lambda < \frac{1}{\sqrt{t}}$ . Thus we get  $(Q - \lambda A)^{T_2} \in \mathcal{W}$ . Now it follows from Theorem 2.1 (1) that  $(Q - \lambda A)^{T_2}$  is finer than  $W$ , a contradiction.  $\square$

**Corollary 3.6.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W \in \mathcal{W}(H_1 \otimes H_2)$  be a decomposable entanglement witness. If  $W$  is optimal, then  $W^{T_2} \notin \mathcal{W}$ .*

**Proof.** By Theorem 3.4, we know that  $W = Q^{T_2}$  for some  $Q \geq 0$ . Therefore,  $W^{T_2} = Q \geq 0$ .  $\square$

For low dimensional systems, the optimal witnesses are easily constructed. For example, the optimal witnesses for two qubits (i.e., the  $2 \times 2$  system) are of the form

$$W = |\psi\rangle\langle\psi|^{T_2},$$

where  $|\psi\rangle$  is an entangled state vector [13]. In fact, an optimal witness detecting the state  $\rho$  can be constructed from the eigenvector  $|\psi\rangle$  of  $\rho^{T_2}$  with negative eigenvalue  $\lambda$  as  $W = |\psi\rangle\langle\psi|^{T_2}$  since  $\text{Tr}(|\psi\rangle\langle\psi|^{T_2}\rho) = \text{Tr}(|\psi\rangle\langle\psi|\rho^{T_2}) = \lambda < 0$  [13]. This method can be generalized to infinite-dimensional case but the resulting witness may be not an optimal one.



## 4. WITNESSES WITHOUT THE FINER RELATION BETWEEN THEM

Now we turn back to the question when different entanglement witnesses without “finer” relation between them can detect some entangled states simultaneously. This question was studied in [21] for finite-dimensional cases, there [21, Theorem 4] asserts that, in finite-dimensional systems, under the condition  $\text{Tr}(W_1) = \text{Tr}(W_2)$ , if there exists no inclusion relation between  $\mathcal{D}_{W_1}$  and  $\mathcal{D}_{W_2}$ , then  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$  if and only if  $W = \lambda W_1 + (1 - \lambda)W_2$  is not a positive operator for all  $0 \leq \lambda \leq 1$ . We point out, though the result is true, the proof of [21] is not correct.

Our attention is main focus on the infinite-dimensional cases. We generalize the above result without the assumption “ $\text{Tr}(W_1) = \text{Tr}(W_2)$ ” and provide a proof that valid for both finite-dimensional systems and infinite-dimensional systems.

The following two lemmas are obvious.

**Lemma 4.1.** *Let  $H_1, H_2$  be complex Hilbert spaces and let  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$  with  $W_1 < W_2$ . If  $W(a, b) = aW_1 + bW_2$ ,  $a$  and  $b$  are positive numbers, then  $W_1 < W(a, b) < W_2$ .*

Particularly, if  $W_1 < W_2$ , then all convex combinations of them are still witnesses.

**Lemma 4.2.** *Let  $H_1, H_2$  be complex Hilbert spaces. For  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ , let  $W = aW_1 + bW_2 \neq 0$  with  $a \geq 0$  and  $b \geq 0$ , then  $\mathcal{D}_W \subset \mathcal{D}_{W_1} \cup \mathcal{D}_{W_2}$  and  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \subset \mathcal{D}_W$ .*

The following is our key lemma which is obtained for finite-dimensional cases in [21] with a different and longer proof.

**Lemma 4.3.** *Let  $H_1, H_2$  be complex Hilbert spaces. For  $W, W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ , if  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$  and if  $\mathcal{D}_W \subset \mathcal{D}_{W_1} \cup \mathcal{D}_{W_2}$ , then either  $\mathcal{D}_W \subset \mathcal{D}_{W_1}$  or  $\mathcal{D}_W \subset \mathcal{D}_{W_2}$ .*

**Proof.** Assume, on the contrary, that both  $\mathcal{D}_{W_1} \cap \mathcal{D}_W$  and  $\mathcal{D}_{W_2} \cap \mathcal{D}_W$  are nonempty. Take  $\rho_i \in \mathcal{D}_{W_i} \cap \mathcal{D}_W$ ,  $i = 1, 2$ . Consider the segment  $[\rho_1, \rho_2] = \{\rho_t = (1 - t)\rho_1 + t\rho_2 : 0 \leq t \leq 1\}$ . As  $\mathcal{D}_W$  is convex, we have

$$[\rho_1, \rho_2] \subseteq \mathcal{D}_W \subseteq \mathcal{D}_{W_1} \cup \mathcal{D}_{W_2}.$$

Thus we get

$$[\rho_1, \rho_2] = (\mathcal{D}_{W_1} \cap [\rho_1, \rho_2]) \cup (\mathcal{D}_{W_2} \cap [\rho_1, \rho_2]),$$

that is,  $[\rho_1, \rho_2]$  is divided into two convex parts. It follows that there is  $0 < t_0 < 1$  such that  $\{\rho_t : 0 \leq t < t_0\} \subseteq \mathcal{D}_{W_1}$ ,  $\{\rho_t : t_0 < t \leq 1\} \subseteq \mathcal{D}_{W_2}$ , and either  $\rho_{t_0} \in \mathcal{D}_{W_1}$  or  $\rho_{t_0} \in \mathcal{D}_{W_2}$ . Assume that  $\rho_{t_0} \in \mathcal{D}_{W_1}$ ; then  $\text{Tr}(W_1 \rho_{t_0}) < 0$ . Thus, for sufficient small  $\varepsilon > 0$  with  $t_0 + \varepsilon \leq 1$ , we have

$$0 \leq \text{Tr}(W_1 \rho_{t_0 + \varepsilon}) = \text{Tr}(W_1 \rho_{t_0}) + \varepsilon(\text{Tr}(W_1 \rho_2) - \text{Tr}(W_1 \rho_1)) < 0,$$

a contradiction. Similarly,  $\rho_{t_0} \in \mathcal{D}_{W_2}$  leads to a contradiction, too. This completes the proof.

□

Now we are ready to state and prove the main result in this section, which asserts that two entanglement witnesses without “finer” relation between them can detect some entangled states simultaneously if and only if all convex combinations of them are entanglement witnesses.

**Theorem 4.4.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ . Then  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$  if and only if there exists  $0 < \lambda < 1$  such that  $W = \lambda W_1 + (1 - \lambda)W_2 \geq 0$ .*

By Lemma 4.1, Lemma 4.2 and Theorem 4.4, the following result is immediate, which states that two witnesses can detect some entangled states simultaneously if and only if their convex combination does not break the witness.

**Theorem 4.5.** *Let  $H_1, H_2$  be complex Hilbert spaces and  $W_1, W_2 \in \mathcal{W}(H_1 \otimes H_2)$ . Then  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset$  if and only if  $W_\lambda = \lambda W_1 + (1 - \lambda)W_2 \in \mathcal{W}(H_1 \otimes H_2)$  for all  $0 \leq \lambda \leq 1$ .*

**Proof of Theorem 4.4.** If  $W = \lambda W_1 + (1 - \lambda)W_2 \geq 0$  for some  $\lambda \in (0, 1)$ , then, by Lemma 4.2,  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \subseteq \mathcal{D}_W = \emptyset$ .

Assume that  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} = \emptyset$ . Let  $W(\lambda) = \lambda W_1 + (1 - \lambda)W_2$ ,  $0 \leq \lambda \leq 1$ . Then, by Lemma 4.3, for all  $\lambda \in [0, 1]$ , we have

$$\mathcal{D}_{W(\lambda)} \subset \mathcal{D}_{W_1}, \text{ or } \mathcal{D}_{W(\lambda)} \subset \mathcal{D}_{W_2}.$$

When  $\lambda$  varies from 0 to 1 continuously,  $\mathcal{D}_{W(\lambda)}$  also varies from  $\mathcal{D}_{W_2}$  to  $\mathcal{D}_{W_1}$  continuously. Taken  $\lambda_0 = \sup\{\lambda : \mathcal{D}_{W(\lambda)} \subset \mathcal{D}_{W_2}\}$ .

We claim that, if  $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_2}$  then there exist  $0 < \varepsilon < 1 - \lambda_0$  such that  $W(\lambda_0 + \varepsilon)$  is a positive operator. Otherwise, if for all  $0 < \varepsilon < 1 - \lambda_0$ ,  $\mathcal{D}_{W(\lambda_0 + \varepsilon)} \neq \emptyset$ , then we have

$$\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_2}, \mathcal{D}_{W(\lambda_0 + \varepsilon)} \subset \mathcal{D}_{W_1},$$

and for all  $\rho \in \mathcal{D}_{W(\lambda_0)}$ , we have

$$\text{Tr}(W(\lambda_0)\rho) < 0, \text{Tr}(W(\lambda_0)\rho) + \varepsilon[\text{Tr}(W_1\rho) - \text{Tr}(W_2\rho)] \geq 0.$$

Noticing that  $\text{Tr}(W_1\rho) \geq 0$  and  $\text{Tr}(W_2\rho) < 0$ , the second part of the last inequality is positive, and  $\varepsilon$  is an arbitrarily small positive number, hence the last inequality is impossible. (We remark that there is a mistake in the proof of [21, Theorem 4] right here. In [21], the argument is “for all  $\rho \in \mathcal{D}_{W(\lambda_0 + \varepsilon)}$ , we have

$$\text{Tr}(W(\lambda_0)\rho) \geq 0, \text{Tr}(W(\lambda_0)\rho) + \varepsilon[\text{Tr}(W_1\rho) - \text{Tr}(W_2\rho)] = \text{Tr}(W(\lambda_0 + \varepsilon)\rho) < 0.$$

Noticing that  $\text{Tr}(W_1\rho) < 0$  and  $\text{Tr}(W_2\rho) \geq 0$ , the second part of the last inequality is negative, and  $\varepsilon$  is an arbitrarily small positive number, hence the last inequality is impossible.” However,  $\text{Tr}(W(\lambda_0)\rho)$  maybe equals 0 for all possible  $\rho$  and the above argument is invalid.)

On the other hand, if  $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_1}$  then there exist  $0 < \varepsilon < \lambda_0$  such that  $W(\lambda_0 - \varepsilon)$  is a positive operator. Otherwise, if for all  $0 < \varepsilon < \lambda_0$ ,  $\mathcal{D}_{W(\lambda_0 - \varepsilon)} \neq \emptyset$ , then we have

$$\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_1}, \mathcal{D}_{W(\lambda_0 - \varepsilon)} \subset \mathcal{D}_{W_2},$$

and for all  $\rho \in \mathcal{D}_{W(\lambda_0)}$ , we have

$$\text{Tr}(W(\lambda_0)\rho) < 0, \text{Tr}(W(\lambda_0)\rho) + \varepsilon[\text{Tr}(W_2\rho) - \text{Tr}(W_1\rho)] \geq 0.$$

Noticing that  $\text{Tr}(W_2\rho) \geq 0$  and  $\text{Tr}(W_1\rho) < 0$ , the second part of the last inequality is positive, and  $\varepsilon$  is an arbitrarily small positive number, hence the last inequality is impossible (We remark that there is a mistake similar to that pointed above in the proof of [21, Theorem 4] right here, too.)

To sum up the discussion above, no matter  $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_1}$  or  $\mathcal{D}_{W(\lambda_0)} \subset \mathcal{D}_{W_2}$  there exists  $\lambda \in [0, 1]$  such that  $W(\lambda)$  is a positive operator, which completes the proof of the theorem.  $\square$

In what follows, we generalize Theorem 4.4 and Theorem 4.5 by allowing of finite many witnesses. The idea of the proof of the statement (1) is similar to that in [21] for finite-dimensional cases. Denote by  $\text{cov}(\Gamma)$  the convex hull of  $\Gamma$ , that is, the convex set generalized by  $\Gamma$ .

**Theorem 4.6.** *Let  $H_1, H_2$  be complex Hilbert spaces. For a set of entanglement witnesses,  $\Gamma = \{W_i : 1 \leq i \leq n\} \subseteq \mathcal{W}(H_1 \otimes H_2)$ . Then*

(1)  $\mathcal{D}_\Gamma = \emptyset$  if and only if  $\text{cov}(\Gamma)$  contains some positive operators.

(2)  $\mathcal{D}_\Gamma \neq \emptyset$  if and only if  $\text{cov}(\Gamma) \subseteq \mathcal{W}(H_1 \otimes H_2)$ .

**Proof.** (1) The sufficient part is clear. In fact, if  $W = \sum_{i=1}^n \lambda_i W_i \geq 0$  for some positive number  $\lambda_i$  with  $\sum_i \lambda_i = 1$ , then  $\mathcal{D}_W = \emptyset$ , which implies that  $\mathcal{D}_\Gamma = \emptyset$  since  $\mathcal{D}_\Gamma \subseteq \mathcal{D}_W$ .

Conversely, if  $\mathcal{D}_\Gamma = \emptyset$ , we assume, without loss of generality that, any subset of  $\Gamma$  can detect some entangled states simultaneously. If  $n = 2$ , the theorem becomes Theorem 4.4. Assume that the theorem holds for  $k \leq n - 1$ . By induction, we have to show that the theorem holds for  $n$ . Since the method is the same, we only need to show the case  $n = 3$ . By assumption, we have

$$\mathcal{D}_{W_1} \neq \emptyset, \mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \neq \emptyset, \mathcal{D}_{W_1} \cap \mathcal{D}_{W_3} \neq \emptyset,$$

but

$$\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2} \cap \mathcal{D}_{W_3} = \emptyset,$$

namely,

$$(\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}) \cap (\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}) = \emptyset.$$

Let

$$W(\lambda) = \lambda W_2 + (1 - \lambda) W_3, \lambda \in [0, 1],$$

then

$$\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda)} \subset (\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}) \cup (\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}).$$

Since  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}$  and  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}$  are disjoint, and  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda)}$  is convex, we know that  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda)}$  varies from  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_3}$  to  $\mathcal{D}_{W_1} \cap \mathcal{D}_{W_2}$  whenever  $\lambda$  varies from 0 to 1. Using

the similar argument as that in the proof of Theorem 4.4, we can conclude that there exists  $0 < \lambda_0 < 1$  such that

$$\mathcal{D}_{W_1} \cap \mathcal{D}_{W(\lambda_0)} = \emptyset.$$

Therefore,

$$W = \mu W_1 + (1 - \mu)W(\lambda_0) = \mu W_1 + (1 - \mu)\lambda_0 W_2 + (1 - \mu)(1 - \lambda_0)W_3 \geq 0$$

for some  $\mu \in (0, 1)$ . By induction on  $n$ , we complete the proof of (1).

(2) The “only if” part is obvious. To check the “if” part, assume that  $\text{cov}(\Gamma) \subseteq \mathcal{W}(H_1 \otimes H_2)$ . If, on the contrary,  $\mathcal{D}_\Gamma = \emptyset$ , then, by the statement (1) just proved above, there exists  $W \in \text{cov}(\Gamma)$  such that  $W \geq 0$ . It follows that  $W \notin \mathcal{W}$ , a contradiction.  $\square$

By Theorem 4.6 it is clear that  $W_1, \dots, W_n \in \mathcal{W}(H_1 \otimes H_2)$  detect some entangled states simultaneously if and only if all convex combinations of them are witnesses.

## 5. CONCLUSION

To sum up, in this paper, we answer the question under what conditions different witnesses may detect some entangled states simultaneously. Generally speaking, for bipartite quantum systems, finite many different witnesses can detect some entangled states simultaneously if and only if their convex combinations are still witnesses; they can not detect any entangled state simultaneously if and only if the set of their convex combinations contains a positive operator. For two witnesses  $W_1$  and  $W_2$ , some more can be said: (1)  $W_1$  and  $W_2$  can detect the same set of entangled states if and only if they are linearly dependent; (2)  $W_2$  can detect more entangled states than that  $W_1$  can if and only if  $W_1$  is a linear combination of  $W_2$  and a positive operator. As an application of above results, we show that a witness is optimal if and only if any negative permutation of it will break the witness, that is, a witness  $W$  is optimal if and only if  $W - D$  is not a witness for any positive operator  $D$ ;  $W$  is decomposable optimal implies that  $W$  is a partial transpose of some positive operator.

Before the end, we would like to stress that our results holds for both infinite-dimensional and finite-dimensional cases. Though some of them are known for finite-dimensional systems under the additional assumption  $\text{Tr}(W_1) = \text{Tr}(W_2)$ , the proof of our results are quite different.

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